Lecture 27 (11/03/2021)

* Prayer
* Spiritual thought

* Laplace transform of piecewise functions:

Ex

$$f(t) = \begin{cases} 2 & \text{if } t>1 \\ -1 & \text{if } t<1 \end{cases}$$

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-ts}dt = \int_0^1 -e^{-ts}dt + \int_1^\infty 2e^{-ts}dt$$

$$= \frac{1}{s}e^{-ts}\Big|_{t=0}^{t=1} + \frac{2}{-s}e^{-ts}\Big|_{t=1}^{t\to\infty}$$

$$= \frac{1}{s}(e^{-s}-1) - \frac{2}{s}(0-e^{-s})$$

$$= \frac{3e^{-s}-1}{s}$$

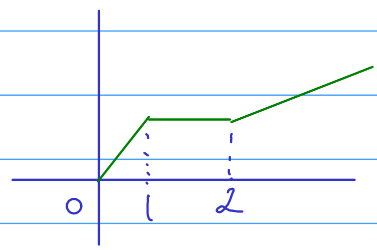
* Any piecewise function can be written in terms of step functions:

$$u_c(t) = \begin{cases} 1 & \text{if } t>c \\ 0 & \text{if } t<c \end{cases}$$

Heaviside function

Ex:

$$f(t) = \begin{cases} t & \text{if } t<1 \\ 1 & \text{if } 1<t<2 \\ t/2 & \text{if } t>2 \end{cases}$$



$$f(t) = t\,u_0(t) + (1-t)u_1(t) + \left(\frac{t}{2}-1\right)u_2(t)$$

Ex

$$f(t) = \begin{cases} 1 & \text{if } t<2 \\ t+2 & \text{if } 2<t<3 \\ e^t & \text{if } 3<t<4 \\ \sin t & \text{if } t>4 \end{cases}$$

$$= u_0(t) + (t+1)u_2(t) + (e^t-t-2)u_3(t) + (\sin t - e^t)u_4(t)$$

It is easy to check that $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f\}$,

Ex:

Find Laplace transform of the function

$$f(t) = \begin{cases} t & \text{if } t<1 \\ 1 & \text{if } 1<t<2 \\ t/2 & \text{if } t>2 \end{cases}$$

Recall: $f(t) = \underbrace{t}_{f_1(t-0)} u_0(t) + \underbrace{(1-t)}_{f_2(t-1)} u_1(t) + \underbrace{\left(\frac{t}{2}-1\right)}_{f_3(t-2)} u_2(t)$

$$f_1(t) = t, \quad f_2(t) = -t, \quad f_3(t) = t/2$$

$$\mathcal{L}\{f\} = \underbrace{e^{-0s}}_{=1} \mathcal{L}\{f_1\} + e^{-s}\mathcal{L}\{f_2\} + e^{-2s}\mathcal{L}\{f_3\}$$

$$= \frac{1}{s^2} - e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{2}\frac{1}{s^2}$$

$$= \frac{2-2e^{-s}+e^{-2s}}{2s^2}$$

* Solve the mass-spring vibration:

$$\begin{cases} y''+2y'+2y = f(t) \\ y(0)=0, \ y'(0)=0 \end{cases}$$

where $f(t) = \begin{cases} \cos t, & 0 \le t \le 10\pi \\ 0, & t > 10\pi \end{cases}$

We can rewrite f as $f(t) = (\cos t) u_0(t) - (\cos t) u_{10\pi}(t)$